Static and dynamic epidemics on looped chains and looped trees

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The dynamic epidemic model [N. Vandewalle and M. Ausloos, J. Phys. A **29**, 309 (1996)] considers the growth of a cluster in a medium containing a fraction x of mobile ''particles'' that are pushed by a propagation front. This model is exactly solved here on various chains and trees that contain loops following an ''evolution matrix'' method. The exact value for the percolation threshold x_c and the critical exponents are calculated for static and mobile particles, respectively. Surprisingly, the mobile character of the particles affects the values of the critical exponents on chains but not on trees. Thus there is a nonuniversal behavior for dynamic epidemics even on $d=1$ lattices. [S1063-651X(96)06910-3]

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I. INTRODUCTION

In order to study the case of a far from equilibrium invasion of a phase in a random medium, various simple models have been imagined such as invasion percolation [1], diffusion-controlled reaction [2], forest fire [3], and epidemic models [4]. The percolation phenomenon theory generally gives a useful framework for studying these models. One usually searches for critical thresholds and critical exponents near percolation thresholds. Most of time, the critical exponent values depend only on the dimensionality of the underlying lattice.

Less attention has been paid to the cases in which a dynamical interaction exists between the spreading phase and the medium. One can cite as an interesting example the case of particles that are pushed by a solid-liquid interface [5]. Related to this kind of problem, let us mention bubbles [6] and biological cells [7] pushed by ice-water fronts and incorporation of secondary phases in type-II superconductors [8]. Such a multiparticle problem is of great interest because of the presence of impurities that are finally trapped. The impurities can influence the physical properties of the medium. For example, the Y_2BaCuO_5 particles left after an incomplete peritectic reaction act as pinning centers for the vortex motion in the $YBa_2Cu_3O_{7-\delta}$ superconducting ceramics [9].

Recently, we have introduced a dynamic epidemic model [10] for describing such a spreading of a ''fluid'' in a medium containing a fraction x of mobile particles. This model assumes a repulsive interaction between the growing front and the particles. The pushing of such particles leads to a nontrivial clustering of particles along (and behind) the growing front [11]. The process sometimes leads to the trapping of particle aggregates behind the front as in an engulfment process when the front overcomes the impurities. On the square lattice, a transition between unlimited and limited growth was found to take place at some particle fraction $x_c=0.560\pm0.005$, i.e., much larger than the threshold value $x_c^{(e)}\sim0.413$ of the epidemic (static particle) model. Our previous work [10] on the square lattice underlined the different origins for the transition in the ''dynamic epidemic'' and in the ''static epidemic'' model. Surprisingly, the critical exponents for the transition with mobile particles were, however, found to be equivalent to those of two-dimensional percolation as for the case of an invaded static medium [12].

In a recent theoretical work [13], we solved exactly this dynamic epidemic model on the usual Bethe lattice for any branching rate z. The Bethe lattice corresponds in some sense to an infinite dimensionality network. Thus we found a percolation threshold $x_c^{(dyn)}=(z^2-1)/z^2$ different from that of the static case, i.e., $x_c^{(sta)}=(z-1)/z$. We found that critical exponents on the Bethe lattice for the static and the dynamic medium problem were equivalent. This suggested that the phenomenology of the phenomenon was well described by our theoretical developments [13].

However, one should remark that the usual Bethe lattice contains no loop as in ordinary lattices. Therefore, particles cannot be trapped by the growing cluster on the Bethe lattice. The aim of this paper is to provide an exact solution of the static and dynamic epidemic model on various lattices with loops thus allowing for trapping by engulfment. Chains with loops and looped trees, i.e., trees for which a branch (or bond) is replaced by a loop (a triangle or a square), are considered below. Both lattices correspond to the one- and infinite-dimensional cases, respectively. Notice that the infinite dimensionality of trees is only asymptotic as discussed in [12] by Stauffer and Aharony for the case of the Bethe lattice. To our knowledge, it is the first time that ''looped trees'' are considered in theoretical developments for percolationlike studies.

In the next section the case of unlooped and looped chains will be explicitly solved as an illustration of the theoretical technique used in this paper. In Sec. III the theoretical results will be presented for various looped trees. We will discuss results in Sec. IV before drawing a conclusion in Sec. V.

II. STATIC AND DYNAMIC EPIDEMICS ON UNLOOPED AND LOOPED CHAINS

As an illustration of the technique used herein, we will solve in this section the simplest chain cases. In Figs. 1(a)–1(c) three different chains are presented: (a) the first one is a conventional chain, (b) the second one is a looped chain with

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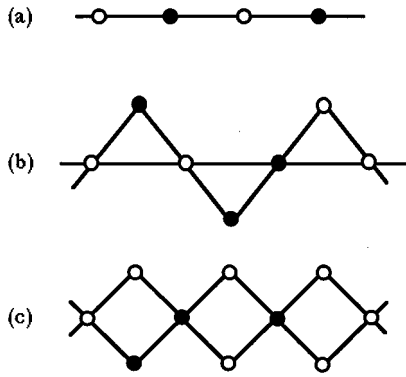†Electronic address: ausloos@gw.unipc.ulg.ac.be

FIG. 1. Three different types of chains: (a) a simple chain, (b) a chain of triangles, and (c) a chain of squares. The empty sites are denoted by white dots, while the sites occupied by a particle are denoted by black dots.

triangles, and (c) the third one is a looped chain with squares. The empty sites are denoted by white dots, while the sites occupied by a particle are denoted by black dots. One should note that the fraction φ of sites that are candidates for particle trapping takes different values on these lattices. For the simple chain case, the fraction φ is zero. However, the chain looped with triangles [Fig. 1(b)] has $\varphi = \frac{1}{2}$, while the chain looped with squares [Fig. 1(c)] has $\varphi = \frac{2}{3}$ because particles can be trapped behind the propagation front.

A. Simple chain

On a simple one-dimensional chain, the dynamic epidemic rule is the following. Let a fraction x of sites be occupied by particles and for convenience let the front go from left to right. The cluster growth goes as follows. At each growth step, if a nearest neighboring site on the right is empty, the site is invaded. However, if this site contains a particle, the particle is pushed on the next-nearest-neighboring site if the latter is empty. If this next-nearest-neighboring site contains a particle, the growth is blocked. In this case, the invading cluster cannot percolate throughout the whole chain.

In order to describe this dynamical process of cluster spreading and particle pushing, it is sufficient to consider a set of four local configurations (Fig. 2). If the invading cluster arrives on the left of the configurations, it could invade the first three configurations of Fig. 2. However, the last
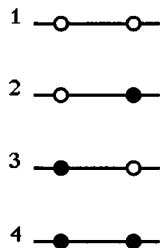

FIG. 2. Four local configurations used to describe the dynamical process for the epidemics on a simple chain. The empty sites are denoted by white dots, while the sites occupied by a particle are denoted by black dots.

configuration (4) blocks the growth since the particle on the left of the configuration cannot move. Moreover, for configuration 3, the invasion on the left site pushes the particle on the empty next-nearest-neighboring site, leading to a 2 configuration. The dynamical process corresponding to the invasion of the left-hand site is thus described by the 4×4 transfer matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (1)$$

where each element T_{ij} is the probability to obtain the configuration labeled i invading the configuration labeled j such that $|i\rangle = \Sigma_j T_{ij}|j\rangle$. The last column of T contains only zeros since this configuration does not evolve. Once the dynamical process (T) is applied, one has to know the probability to find the next configuration on the chain. This is simply given by the growth probability matrix

$$G = \begin{pmatrix} 1-x & 0 & 1-x & 0 \\ x & 0 & x & 0 \\ 0 & 1-x & 0 & 1-x \\ 0 & x & 0 & x \end{pmatrix}, \qquad (2)$$

where each element G_{ij} is the average number of configurations i next to a configuration j. The growth process is thus well described by the iteration of the matrix GT. This matrix has two degenerate nonzero eigenvalues λ, i.e.,

$$\lambda = (1-x). \qquad (3)$$

The critical point $x_c^{(\mathrm{dyn})}$ is when the largest eigenvalue is equal to 1. The spreading of the invading cluster through the chain is thus only possible in the absence of impurities ($x_c^{(\mathrm{dyn})} = 0$). This is similar to the trivial case of static particles, i.e., to percolation on a simple chain [12]. One should note that the case of static particles is recovered for a T matrix that is reduced to

$$T^{(\mathrm{sta})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (4)$$

such that the $GT^{(\mathrm{sta})}$ matrix has only one nonzero eigenvalue $\lambda = 1-x$ giving $x_c^{(\mathrm{sta})} = 0$.

The above development is somewhat trivial on a simple chain but can be applied to more elaborate lattices. We solve the epidemic spreading on looped chains (Fig. 1) next.

B. Chain of triangles

Figure 3 presents the eight local configurations for the description of the epidemic propagation on the chain with triangle loops [Fig. 1(b)]. Each basic configuration contains three sites, the one at the top being the candidate on which the particle trapping can take place. The invasion of the local configurations corresponds to growth steps depending on the local configurations, i.e., on the number of particles lying on the local configuration.
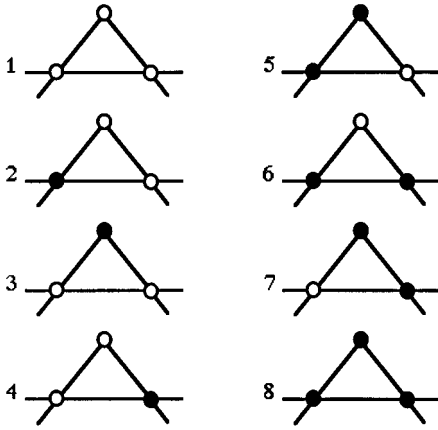
FIG. 3. Eight local configurations used to describe the dynamic epidemic model on a chain of triangles. The empty sites are denoted by white dots, while the sites occupied by a particle are denoted by black dots.

All possibilities will be hereby enumerated for this particular lattice in order to illustrate how the T matrix is obtained in general.

(i) Let the epidemics invade the left-hand site of the configuration 1. Once this site is invaded, the spreading cluster will invade either the "top site" (see Fig. 3) with a probability $\frac{1}{2}$ or the "right-hand site" with a probability $\frac{1}{2}$ at the next step. In both cases, configuration 1 remains unchanged by the dynamical invasion, whence $T_{11}=1$.

(ii) Let the epidemics invade configuration 2 by pushing the particle either to the top site or to the right-hand site both with a probability $\frac{1}{2}$. However, two situations are possible after this step. The particle pushed on the top site can be trapped by engulfment if the invasion front overpasses the particle. This occurs with a probability $\frac{1}{4}$ $(T_{32}=\frac{1}{4})$. But the particle can be pushed to the right-hand site in the next step. This second final configuration occurs with a probability $\frac{3}{4}$ $(T_{42}=\frac{3}{4})$.

(iii) A two-stage dynamic invasion as for configuration 3 can also lead to particle trapping on the top site. This engulfment occurs with a probability $\frac{1}{2}$ $(T_{33}=\frac{1}{2})$, while the particle pushing to the right-hand site occurs with a probability $\frac{1}{2}$ $(T_{43}=\frac{1}{2})$.

(iv) The two-step dynamic invasion of configuration 4 does not affect this local configuration since it leaves unchanged the particle on the right-hand site. Thus $T_{44}=1$.

(v) Let the particle lying on the left-hand site of configuration 5 being pushed to the right-hand site by the invading cluster. The other particle lying on the top site remains there since the right-hand site becomes occupied. This leads to $T_{75}=1$.

(vi) The particle lying on the left-hand site of configuration 6 can be pushed only to the top site, since the right-hand site is occupied. This leads to the stable configuration 7 and $T_{76}=1$.

(vii) In configuration 7 the epidemics can invade the left-hand site, but the epidemics cannot invade the top site since the particle lying there is blocked. Thus $T_{77}=1$.

(viii) Configuration 8 cannot be invaded. Thus the last column of T contains only zeros.

The T and G matrices respectively representing the local dynamical processes on the local configurations and the probability to obtain the next ones are thus given by

$$T=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

and

$$G=\begin{pmatrix} (1-x)^2 & (1-x)^2 & (1-x)^2 & 0 & (1-x)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-x)^2 & 0 & (1-x)^2 & (1-x)^2 & (1-x)^2 \\ x(1-x) & x(1-x) & x(1-x) & 0 & x(1-x) & 0 & 0 & 0 \\ x(1-x) & x(1-x) & x(1-x) & 0 & x(1-x) & 0 & 0 & 0 \\ 0 & 0 & 0 & x(1-x) & 0 & x(1-x) & x(1-x) & x(1-x) \\ 0 & 0 & 0 & x(1-x) & 0 & x(1-x) & x(1-x) & x(1-x) \\ x^2 & x^2 & x^2 & 0 & x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^2 & 0 & x^2 & x^2 & x^2 \end{pmatrix}. \tag{6}$$

One should remark that the elements of the matrix T can take noninteger values. This is a characteristic of lattices with loops on which trapping occurs with noninteger probabilities. The sum of the elements of each column of T is 1, expressing the normalization of the probabilities except for the elements of the last column of T, which are zeros. The sum of the elements of each column of G is also 1 since each local configuration is connected to only one configuration on a chain. Notice also that numerous elements of G are zeros. This means that some local configurations cannot be causally connected, in other words, cannot grow.
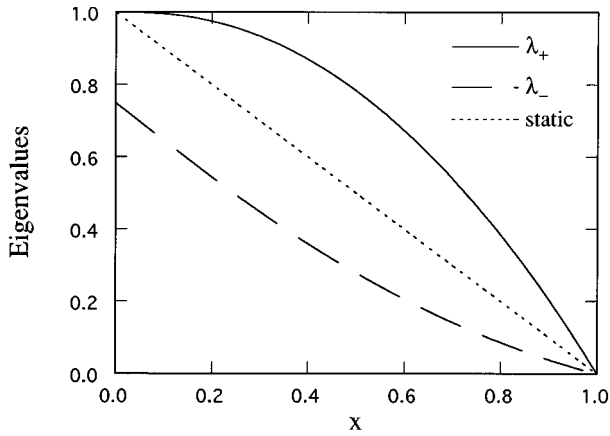
FIG. 4. Two nonzero eigenvalues $\lambda_\pm$ of GT as a function of the fraction x of particles on a chain of triangles. The eigenvalue for the static case is also illustrated.

The GT matrix has two different nonzero eigenvalues $\lambda_\pm$ given by

$$\lambda_\pm = \tfrac{1}{8}[7-4x-3x^2 \pm (1-x)\sqrt{1+10x+41x^2}], \qquad (7)$$

which are both strictly less than 1 except for $x=0$ (Fig. 4). Therefore, the dynamic epidemics can spread throughout the chain only for $x_c^{\mathrm{dyn}}=0$.

One should remark that the $GT^{\mathrm{(sta)}}$ matrix can be easily written. It has only one nonzero eigenvalue $\lambda=1-x$ giving $x_c^{\mathrm{(sta)}}=0$.

The technique applied above is thus useful for describing dynamic epidemics on looped chains. More elaborate looped lattices will lead to a larger number of local configurations. We should warn that the configuration label or order is crucial for an optimal description of the transfer matrices, i.e., in order to have block-diagonal matrices.

C. Chain of squares

On a regular chain of squares, the basic set of local configurations contains four sites (Fig. 5). In order to describe the dynamic epidemics on a chain looped with squares, there are thus $2^4=16$ different possible configurations to examine. However, this set is reduced to 12 configurations by symmetry arguments (Fig. 5).

After some algebra, the GT matrix of size 12×12 is found to have two nonzero eigenvalues $\lambda_\pm$ given by

$$\lambda_\pm = \tfrac{1}{8}[7-7x-7x^2+7x^3 \pm (1-x)^2\sqrt{1+18x+129x^2}]. \qquad (8)$$

The critical fraction of particles is $x_c^{\mathrm{(dyn)}}=0$. One can easily find that the corresponding $GT^{\mathrm{(sta)}}$ for static particles has only one nonzero eigenvalue

$$\lambda = 1-x-x^2+x^3, \qquad (9)$$

which gives a critical threshold $x_c^{\mathrm{(sta)}}=0$. The $\lambda_\pm$ and λ values as a function of x are shown in Fig. 6.

One should emphasize that the λ eigenvalue corresponding to the static case is a nonlinear function of x in contrast to the simple chain and the chain looped with triangles. This
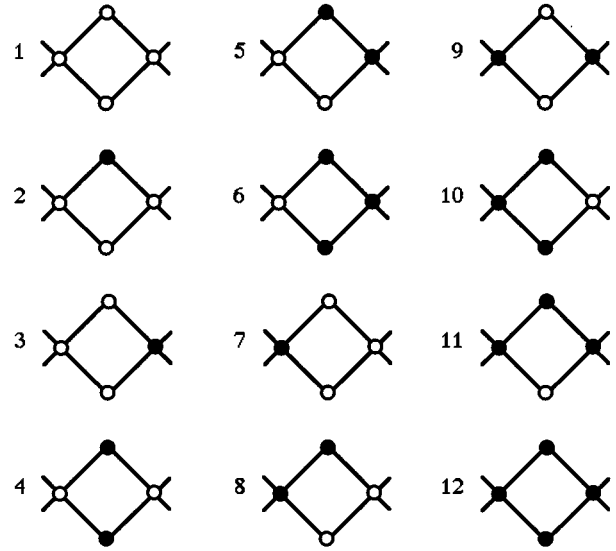

FIG. 5. Twelve local configurations used to describe the dynamic epidemic model on a chain of squares. The empty sites are denoted by white dots, while the sites occupied by a particle are denoted by black dots.

nonlinearity arises from the fact that the epidemics successively invade sites that are geometrically nonequivalent for loop to loop spreading.

D. Critical exponents

One usually observes a power-law divergence for various physical quantities near a percolation threshold, e.g., the linear size ξ of the spreading cluster and the cluster mass S,

$$\xi \sim (x-x_c)^{-\nu} \qquad (10)$$

and

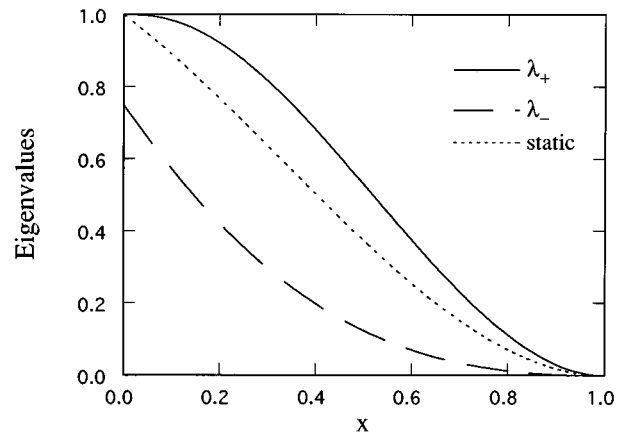$$S \sim (x-x_c)^{-\gamma} \qquad (11)$$

FIG. 6. Two nonzero eigenvalues $\lambda_\pm$ of GT as a function of the fraction x of particles on the chain looped with squares. The eigenvalue for the static case is also illustrated.
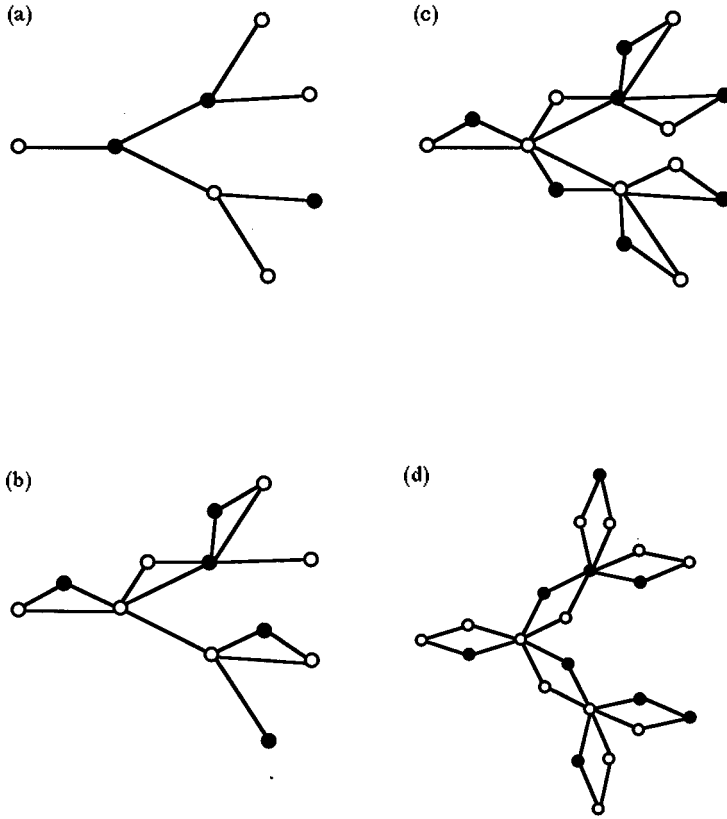
FIG. 7. Four different types of hierarchical trees with a branching rate $z=2$: (a) the Bethe lattice, (b) a tree for which one branch over two is looped with a triangle, (c) a tree of triangles, and (d) a tree of squares.

above x_c. Assuming that the largest eigenvalue drives the growth process, the mean number $g(r)$ of sites lying on the spreading cluster at a distance r from the initial site is

$$g(r) \sim \lambda_+^r . \tag{12}$$

Then the correlation length ξ, defined by

$$\xi^2 = \left(\sum_{r=1}^{+\infty} r^2 g(r) \right) \left(\sum_{r=1}^{+\infty} g(r) \right)^{-1} , \tag{13}$$

gives simply $\xi^2 \sim (1-\lambda_+)^{-2}$ using Eq. (12). The mean mass S of finite spreading clusters (for $x > x_c$) is defined by

$$S = 1 + \sum_{r=1}^{+\infty} g(r), \tag{14}$$

which gives simply $S \sim (1-\lambda_+)^{-1}$ using relation (12).

One can develop $1-\lambda_\pm$ in powers of x around $x_c=0$ in order to obtain the critical exponents ν and γ. Notice that the scaling laws $\xi^2 \sim (1-\lambda_+)^{-2}$ and $S \sim (1-\lambda_+)^{-1}$ lead to $\nu = \gamma$ for chains with or without loops.

For the simple chain, $\nu = \gamma = 1$ for both static and dynamic cases. For the chain of triangles, the transition is characterized by $\nu = \gamma = 1$ in the static case since $\lambda = 1 - x$. The expansion of λ_+ [from Eq. (6)] gives $\nu = \gamma = 3$ for the dynamic epidemics. For the chain looped with squares, the exponents are $\nu = \gamma = 1$ in the static case but $\nu = \gamma = 3$ in the dynamic case.

III. STATIC AND DYNAMIC EPIDEMICS ON UNLOOPED AND LOOPED TREES

A. Introduction

For the usual Bethe lattice with a branching rate z [13], we found a critical particle fraction $x_c^{(\mathrm{dyn})} = (z^2-1)/z^2$ much larger than the static case, where $x_c^{(\mathrm{sta})} = (z-1)/z$. The critical exponents for dynamic epidemics were found to be $\gamma=1$ and $\nu=1$ as in the static case.

On Bethe lattices and on more general trees, a special role is played by the exponent ν defined with Eqs. (10), (12), and (13) [14]. This exponent characterizes the divergence of the correlation length along the tree, also called the chemical space. In a truly Euclidian distance, the coherence length diverges with an exponent $\tilde{\nu} = \nu/2$ above the so-called critical dimension [14].

Since critical exponents for dynamic percolation seem to be sensitive to the presence of loops on a chain, it is of interest to look at trees with loops. For simplicity, we restrict our discussion to $z=2$ trees [Figs. 7(a)–7(d)] with different fractions φ of trapping sites: (a) the $z=2$ Bethe lattice ($\varphi=0$), (b) a tree for which one branch out of two is a triangle ($\varphi = \frac{1}{2}$), (c) a tree of triangles ($\varphi = \frac{2}{3}$), and (d) a tree of squares joining at corners ($\varphi = \frac{4}{5}$). For convenience, we will call I and II the trees looped with triangles having a fraction φ of trapping sites $\frac{1}{2}$ and $\frac{2}{3}$, respectively.

B. Tree looped with triangles

1. Tree of type I

For the tree of triangles shown in Fig. 7(b) ($\varphi = \frac{1}{2}$), the local configurations contain four sites as schematically
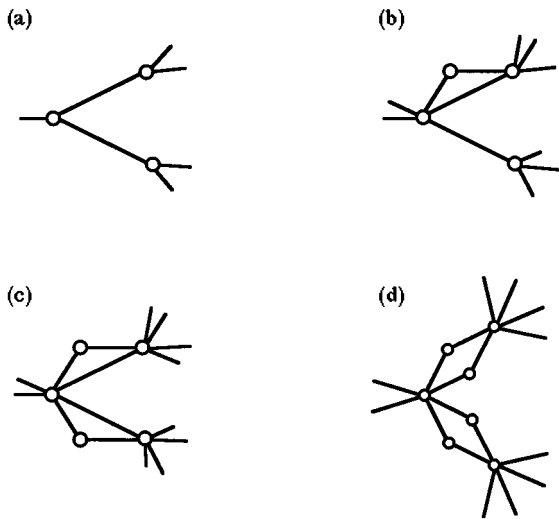
FIG. 8. Small portions of the trees of Fig. 7 that are used as local configurations for studying the dynamic epidemic model: (a) the Bethe lattice, (b) a tree for which one branch over two is looped with a triangle, (c) a tree of triangles, and (d) a tree of squares.

FIG. 9. Two nonzero eigenvalues $\lambda_\pm$ of GT as a function of the fraction x of particles on the tree of triangles of type I [the tree of Fig. 7(b)]. The eigenvalue for the static case is also illustrated.

drawn in Fig. 8(b). In order to describe the dynamic epidemic model on such a tree, there are thus $2^4 = 16$ different possible configurations. One cannot reduce this set of 16 configurations since the local configurations have no symmetry. Similarly to the above theoretical developments, one can, however, enumerate all possible configurations and processes occurring on the tree. Thereafter, one can write the 16×16 T matrix describing the dynamical process and the 16×16 G matrix describing the occurrence of the next local configurations. One should note that the sum of the elements of each column of G is now equal to 2 instead of 1 in Sec. II since each local configuration is followed by $z = 2$ next possible configurations.

Even though it was possible to write down the 16×16 GT matrix, it was not possible to extract analytically the eigenvalues. We found numerically that two different eigenvalues have nonzero values for the dynamic epidemics. As above, we call them λ_+ and λ_- with $\lambda_+ \geq \lambda_-$. These eigenvalues are shown in Fig. 9 as a function of the particle fraction x. We observe that for $x = 0$ the largest eigenvalue λ_+ is 2. In absence of particles, the tree can be totally invaded such that λ_+ corresponds to the branching rate of the tree [see Eq. (12)]. As the mobile particle fraction x increases, the λ_+ eigenvalue decreases down to zero at $x = 1$ when the growth is impossible. We found numerically $x_c^{(\text{dyn})} \approx 0.812$ for such a type of looped tree. Below $x_c^{(\text{dyn})}$, the invasion propagates indefinitely. Above $x_c^{(\text{dyn})}$, the epidemics is blocked.

Moreover, we have numerically analyzed how λ_+ reaches 1 near $x_c^{(\text{dyn})}$. We found that $(1 - \lambda_+)$ scales as $|x - x_c^{(\text{dyn})}|$. Thus $\gamma = \nu = 1$ for the dynamic epidemics on such a tree looped with triangles.

The static case is trivially solved because the T matrix reduces to a diagonal $T^{(\text{sta})}$ matrix. We found analytically that the 16×16 GT matrix has only one nonzero eigenvalue $\lambda = 2(1-x)$, which is also drawn in Fig. 9. Thus $x_c^{(\text{sta})} = \frac{1}{2}$, corresponding to the static epidemic case on the usual Bethe lattice. This gives also $\gamma = \nu = 1$. Thus the critical exponents
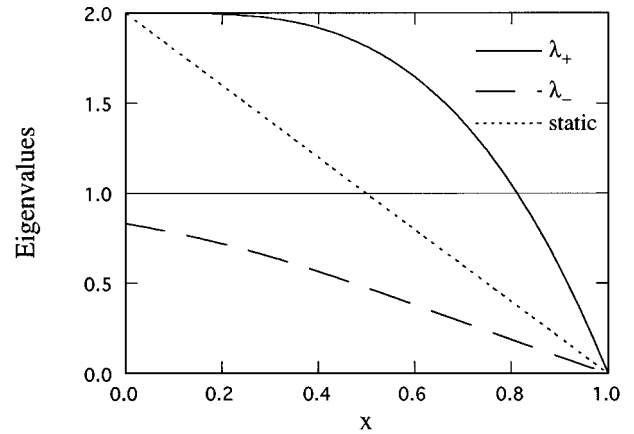
describing the divergence of the cluster mass S and correlation length ξ near the threshold x_c have the same values ($=1$) for both static and dynamic epidemics on a tree looped with triangles of type I.

2. Tree of type II

For the tree made of triangles ($\varphi = \frac{3}{2}$) joining by a vertex as shown in Fig. 7(c), the local basic configurations contain five sites as schematically drawn in Fig. 8(c). In order to describe the dynamic epidemic model on such a tree, there are $2^5 = 32$ different possible configurations. One can reduce this set to 20 configurations by symmetry. One can still enumerate all possible configurations and processes occurring on the tree and write the 20×20 GT matrix. However, it was not possible to extract analytically the eigenvalues of the GT matrix. We found numerically that two different eigenvalues have nonzero values for the dynamic epidemics. Similarly to previous results, we call them λ_+ for the largest one and λ_- for the second one. These eigenvalues are plotted in Fig. 10 as a function of x. As x increases from zero to 1, λ_+ decreases from 2 to zero. We found numerically $x_c^{(\text{dyn})} \approx 0.849$ for looped trees of type II. This value is a little bit larger than
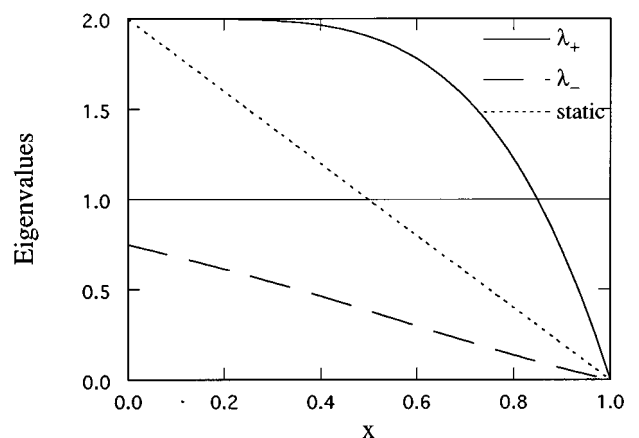

FIG. 10. Two nonzero eigenvalues $\lambda_\pm$ of GT as a function of the fraction x of particles on the tree of triangles of type II [the tree of Fig. 7(c)]. The eigenvalue for the static case is also illustrated.
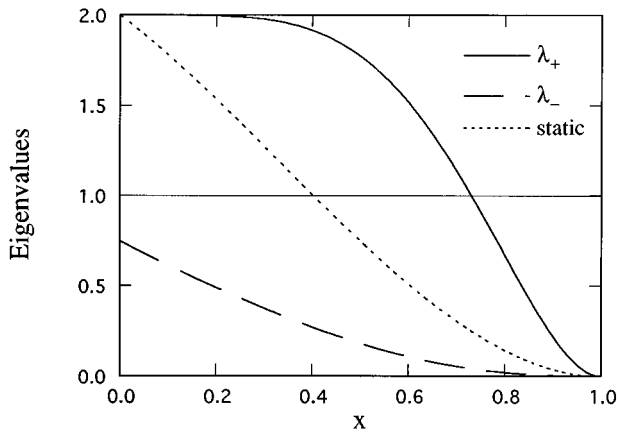
FIG. 11. Two nonzero eigenvalues $\lambda_\pm$ of GT as a function of the fraction x of particles on the tree looped with squares [the tree of Fig. 7(d)]. The eigenvalue for the static case is also illustrated.

the critical value 0.812 found numerically for trees of type I. Moreover, we have numerically analyzed how λ_+ reaches 1 near $x_c^{(dyn)}$. We find $\gamma = \nu = 1$ for the dynamic epidemia on such a tree of triangles.

In the static case, the 20×20 $GT^{(sta)}$ matrix has only one nonzero eigenvalue $\lambda = 2(1-x)$ as drawn in Fig. 10. This eigenvalue is equivalent to that of type I. Thus $x_c^{(sta)} = \frac{1}{2}$ and $\gamma = \nu = 1$.

Thus the critical exponents describing the divergence of the cluster mass S and correlation length ξ near the threshold x_c have the same values ($=1$) for both static and dynamic epidemia on a tree looped with triangles of type II.

C. Tree looped with squares

For a tree made of squares ($\varphi = \frac{4}{5}$) as shown in Fig. 7(d), the local configurations contain seven sites as schematically drawn in Fig. 8(d). The dynamic epidemics on such a tree can be described from $2^7 = 128$ different possible local configurations. One can reduce this set to 42 configurations by symmetry. One can enumerate all possible configurations and dynamical processes occurring on these 42 configurations and write the corresponding 42×42 GT matrix. We found numerically that two different eigenvalues λ_+ and λ_- have nonzero values for the dynamic epidemics (Fig. 11). As the particle fraction increases from zero to 1, λ_+ decreases from 2 to zero. We found numerically that $x_c^{(dyn)} = 0.731$. This threshold is close to the dynamical case on the usual Bethe lattice with a branching $z = 2$. Even though the tree looped with squares contains a larger fraction of trapping sites, i.e., $\varphi = \frac{4}{5}$, than trees looped with triangles, the threshold $x_c^{(dyn)}$ is about 10% lower than that for the trees looped with triangles. We found numerically that $\gamma = \nu = 1$ for the dynamic epidemics on a tree looped with square.

The static case is recovered by reducing the T matrix to $T^{(sta)}$. We found numerically that the 42×42 GT matrix has only one nonzero eigenvalue λ as drawn in Fig. 11. The latter eigenvalue is *not* equivalent to the previous static ones and has a *nonlinear* behavior. A threshold at $x_c^{(sta)} \approx 0.403$ is numerically obtained. A numerical analysis of the behavior of λ near the threshold $x_c^{(sta)}$ gives $\gamma = \nu = 1$. Thus the critical

TABLE I. Percolation thresholds $x_c^{(sta)}$ and $x_c^{(dyn)}$ for, respectively, static and mobile particles on various lattices. For comparison between similar structures, the fraction φ of sites that are possible candidates for trapping is given.

Lattice	d	φ	$x_c^{(sta)}$	$x_c^{(dyn)}$
simple chain	1	0	0^a	0^b
chain of triangles	1	$\frac{1}{2}$	0^b	0^b
chain of squares	1	$\frac{2}{3}$	0^b	0^b
square	2	1	0.413^c	0.565^d
Bethe lattice	∞	0	$(z-1)/z^a$	$(z^2-1)/z^{2\ e}$
tree looped with triangles I	∞	$\frac{1}{2}$	$\frac{1}{2}^b$	0.812^b
tree looped with triangles II	∞	$\frac{2}{3}$	$\frac{1}{2}^b$	0.849^b
tree looped with squares	∞	$\frac{4}{5}$	0.403^b	0.731^b

[a]Exact result; see [12].
[b]Exact result obtained in this paper.
[c]Numerical value obtained in [4].
[d]Numerical value obtained in [10].
[e]Exact result obtained in [15].

exponents describing the divergence of the cluster mass S and correlation length ξ near the threshold x_c have the same values ($=1$) for both static and dynamic epidemics on a tree looped with squares.

IV. DISCUSSION

We have theoretically studied both dynamic and static epidemic problems on chains and generalized trees, i.e., on lattices with dimensionality one and infinity, respectively. On such lattices, loops have been considered allowing for engulfment, i.e., particle trapping. All theoretical results found hereinbefore are summarized in Tables I and II.

The epidemics have been described through a "transfer-growth-matrix method." The largest eigenvalue of the epidemic "evolution matrix" GT is a measure of the correlation length of the growing system [see Eqs. (12) and (13)]. We found that a dynamical interaction between the invasion and the random medium changes the value of the critical threshold above which the invasion is stopped. For static epidemics corresponding to the percolation problem, only one eigenvalue is finite, but it is not a linear function of the particle fraction x for chains and trees looped with squares. The nonlinear effect is understood as resulting from the fact that the epidemics has to invade alternatively nonequivalent sites in order to spread throughout the lattice. For dynamic epidemics, two eigenvalues are finite and both are nonlinear functions of the particle fraction x. This is true for all the lattices examined, except for the simple linear chain case. The nonlinearity is due to particle pushing effects and reflects some organization of the random medium.

The thresholds below which the epidemics percolate have been calculated on the various lattices and for both static and dynamical processes. On all studied chains, the thresholds for both static and dynamic epidemics are equal to zero. However, we found nonzero threshold values on trees. On such structures, we found that the threshold for the dynamic epidemics is much higher than the threshold of the static epidemics ($x_c^{(sta)} < x_c^{(dyn)}$). We observed a threshold differ-

TABLE II. Critical exponents γ and ν describing the divergence of the cluster mass and the correlation length, respectively, near the threshold for static and dynamic epidemics on various lattices.

Lattice	d	Static ν	Static γ	Dynamic ν	Dynamic γ
simple chain	1	1[a]	1[a]	1[b]	1[b]
chain of triangles	1	1[b]	1[b]	3[b]	3[b]
chain of squares	1	1[b]	1[b]	3[b]	3[b]
square	2	$\frac{4}{3}$[a]	$\frac{43}{18}$[a]	1.33[c]	
Bethe lattice	∞	1[a]	1[a]	1[d]	1[d]
tree looped with triangles I	∞	1[b]	1[b]	1[b]	1[b]
tree looped with triangles II	∞	1[b]	1[b]	1[b]	1[b]
tree looped with squares	∞	1[b]	1[b]	1[b]	1[b]

[a]Exact result: see [12].
[b]Exact result obtained in this paper.
[c]Numerical value obtained in [10].
[d]Exact result obtained in [13].

ence of up to 30% in the critical particle fraction between the static and dynamic cases.

In the case of trees looped with triangles, two different structures have been studied. These trees have different fractions φ of sites that are candidates for particle trapping. The percolation thresholds for static epidemics are the same on both types of trees. However, a small difference of about 3% was found between the dynamic epidemic thresholds of both cases. The tree of type II having a larger fraction φ has a larger threshold. The fraction φ of sites candidate for particle trapping on a tree is thus a relevant parameter for the threshold value of a dynamic epidemics. One should note that the tree looped with squares has the largest value $\varphi = \frac{4}{5}$, but has a lower $x_c^{(\mathrm{dyn})}$ value than the trees looped with triangles. It is obvious that the tree structure is relevant for determining the threshold value.

For usual critical phenomena, the threshold values do change from lattice to lattice but the critical exponents depend only on the dimensionality of the considered lattices. In order to test a possible universality for the dynamic epidemics, the critical exponents characterizing respectively the correlation length divergence and the cluster mass divergence near the threshold have been determined. For static epidemics, universality is effectively recovered since the exponents depend only on the lattice dimensionality. However, for dynamic epidemics, critical exponents can take different values in one dimension depending on the looped character of the chains. One should note that the universality of dynamic epidemics is recovered for $d > 1$. Moreover, the values of the critical exponents for $d > 1$ are found to be unchanged between static and dynamic epidemics. The behavior is thus ''superuniversal'' for $d > 1$ in contrast with $d = 1$ lattices.

Thus, for both static and dynamic epidemics, a great variety of behaviors (linearity or nonlinearity of the eigenvalues as a function of x, percolation threshold values, and critical exponents) has been found here depending on nonuniversal parameters such as the looped character of the lattice, the fraction of trapping sites φ, and the structure of the loop itself.

V. CONCLUSION

It should be pointed out that, to our knowledge, this is the first time that *looped trees* are considered in theoretical developments for percolationlike studies. The major finding is the lack of universality for dynamic epidemics on one-dimensional lattices. We found that the critical exponents γ and ν depend on the looped or unlooped character of the chain. This result was unexpected. On trees, γ and ν are not affected by the introduction of a dynamical interaction and loops.

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